Quaternion Octonion Reformulation of Quantum Chromodynamics

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June 30, 2010

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Abstract

We have made an attempt to develop the quaternionic formulation of Yang – Mill's field equations and octonion reformulation of quantum chromo dynamics (QCD). Starting with the Lagrangian density, we have discussed the field equations of $SU(2)$ and $SU(3)$ gauge fields for both cases of global and local gauge symmetries. It has been shown that the three quaternion units explain the structure of Yang- Mill's field while the seven octonion units provide the consistent structure of $SU(3)_C$ gauge symmetry of quantum chromo dynamics.

Key Words: Quaternion, Octonions, Quantum Chromodynamics

PACS No.: 14.80 Hv.

1 Introduction

The role of number system (hyper complex number) is an important factor for understanding the various theories of physics from macroscopic to microscopic level. In elementary particle physics , electromagnetism , the strong and weak nuclear forces are described by a combination of relativity and quantum mechanics called relativistic quantum field theory. The electroweak and strong interactions are described by the Standard Model (SM). Standard Model unifies the Glashow – Salam – Weinberg (GSW) electroweak theory and the quantum chromodynamics (QCD) theory of strong interactions. According to celebrated Hurwitz theorem [1] there exits four-division algebra consisting of $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) [2, 3] and $\mathcal{O}$ (octonions) [4, 5, 6]. All four algebras are alternative with antisymmetric associators. Real number explains will the classical Newtonian mechanics,complex number play an important role for the explanation beyond the framework of quantum theory and relativity. Quaternions are having relations with Pauli matrices explain non abelian gauge theory. Quaternions were very first example of hyper complex numbers having the significant impacts on mathematics & physics. Because of their beautiful and unique properties quaternions attracted many to study the laws of nature over the field of these numbers. Yet another complex system i.e; Octonion may play an important role [6, 7, 8, 9] in understanding the physics beyond strong interaction between color degree of freedom of quarks and their interaction. Quaternions naturally unify [10] electromagnetism and weak force, producing the electroweak $SU(2)\times U(1)$ sector of standard model. Octonions are used for unification programme for strong interaction with successful gauge theory of fundamental interaction i.e; octonions naturally unify [11] electromagnetism and weak force producing $SU(3)_c \times SU(2)_w \times U(1)_Y$. In this paper, we have made an attempt to develop the quaternionic formulation of Yang – Mill's field equations and octonion reformulation of quantum chromo dynamics (QCD). Starting with the Lagrangian density, we have discussed the field equations of $SU(2)$ and $SU(3)$ gauge symmetries in terms of quaternions and octonions. It has been shown that the three quaternion units explain the structure of Yang- Mill's field while the seven octonion units provide the consistent structure of $SU(3)_C$ gauge symmetry of quantum chromodynamics (QCD) as they have connected with the well known $SU(3)$ Gellmann λ matrices. In this case the gauge fields describe the potential and currents associated with the generalized fields of dyons particles carrying simultaneously the electric and magnetic charges.

2 Quaternionic Lagrangian Formulation

Let us consider that we have two spin 1/2 fields, ψ_a and ψ_b. The Lagrangian without any interaction is thus defined [12] as

$$L = [i\overline{\psi}_a\gamma^\mu\partial_\mu\psi_a - m\overline{\psi}_a\psi_a] + [\ \ i\overline{\psi}_b\gamma^\mu\partial_\mu\psi_b - m\overline{\psi}_b\psi_b\ \ ] \tag{1}$$

where m is the mass of particle, $\overline{\psi}_a$ and $\overline{\psi}_b$ are respectively used for the adjoint representations of ψ_a and ψ_b and the γ matrices are defind as

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} (\forall\, j = 1, 2, 3). \tag{2}$$

Here σ_j are the well known 2×2 Pauli spin matrices. Lagrangian density (1) is thus the sum of two Lagrangians for particles a and b. We can write above equation more compactly by combining ψ_a and ψ_b into two component column vector;

$$\psi \ = \ \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{3}$$

and accordingly, there is the adjoint spinor

$$\overline{\psi} \ = \ \begin{pmatrix} \overline{\psi_a} & \overline{\psi}_b \end{pmatrix} \tag{4}$$

where the spinor field ψ, is the described [13] as a quaternion

$$\psi = \psi_0 + e_k\psi_k \quad (\forall\, k = 1, 2, 3) \tag{5}$$

followed by a multiplication rule

$$e_j e_k = -\delta_{jk} + \ \ \epsilon_{jkl} \ \ e_l \ (\forall j, k, l = 1, 2, 3). \tag{6}$$

Here δ_{jk} and ϵ_{jkl} are respectively denoted as Kronecker delta symbol and three index Levi - Civita symbols with their usual definitions.The quaternion conjugates of quaternion basis elements as

$$e_k^\dagger \ = \ e_k. \tag{7}$$

Accordingly the adjoint spinor $\overline{\psi} = \psi^{\dagger}\gamma_0$ ($\psi^{\dagger}$ denotes the Hermitian conjugate spinor) is described as

$$\overline{\psi} = \quad \psi_0 - \quad e_k\psi_k \tag{8}$$

whereas a spinor (3) is described as a quaternion

$$\psi = \psi_0 + e_1\psi_1 + \quad e_2\psi_2 + \quad e_3\psi_3 \tag{9}$$

which can be decomposed as

$$\psi = (\psi_0 + e_1\psi_1) + e_2(\psi_2 - e_1\psi_3) = \psi_a + e_2\psi_b. \tag{10}$$

This is the simpletic representation of quaternions in terms of complex number representations. In equation (10), we have written $\psi_a = (\psi_0 + e_1\psi_1)$ and $\psi_b = (\psi_2 - e_1\psi_3)$ described in terms of the field of real number representations. Accordingly, we may write

$$\overline{\psi} = \psi_0 - e_1\psi_1 - e_2\psi_2 - e_3\psi_3 = (\psi_0 - e_1\psi_1) - e_2\left(\psi_2 - e_1\psi_3\right) = \psi_a^{\dagger} - e_2\psi_b^{\dagger}. \tag{11}$$

So, we may write the quaternionic form of the Lagrangian in terms of ψas

$$L = \quad [i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \quad -m\,\overline{\psi}\psi] \tag{12}$$

where $m = \begin{bmatrix} m_1 & 0 \\ o & m_2 \end{bmatrix}$ is the mass matrix with m_1 is the mass of the field ψ_1 whereas m_2 is that of the field ψ_2.

3 Quaternionic Dirac Equation

Substituting the values of ψ and $\overline{\psi}$ from equations (3) and (4) in equation (12), we get

$$L = i\begin{pmatrix} \overline{\psi_a} & \overline{\psi_b} \end{pmatrix}\gamma^{\mu}\partial_{\mu}\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} - m\begin{pmatrix} \overline{\psi_a} & \overline{\psi_b} \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{13}$$

which is reduced to equation (1). Defining the Eular Lagrangian equation as

$$\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\psi)}\right) = \frac{\partial L}{\partial\psi} \tag{14}$$

and taking the variation with respect to $\overline{\psi_a}$ and $\overline{\psi_b}$, we get

$$i\gamma^{\mu}(\partial_{\mu}\psi_a) - m\,\psi_a = 0 \tag{15}$$

and

$$i\gamma^{\mu}(\partial_{\mu}\psi_b) - m\,\psi_b = 0. \tag{16}$$

Equations (15) and (16) are respectively recalled as the Dirac equations [13] for the spinors $\psi_a$and ψ_b. Similarly if we take the variations with respect to ψ_a and ψ_b we get

$$i\,(\partial_{\mu}\overline{\psi}_a)\gamma^{\mu} + m\overline{\psi}_a = 0 \tag{17}$$

and

$$i\,(\partial_{\mu}\overline{\psi}_b)\gamma^{\mu} + m\overline{\psi}_b = 0 \tag{18}$$

which are respectively recalled as the Dirac equations for the adjoint spinors $\overline{\psi_a}$ and $\overline{\psi_b}$. In equations (15-18) the γ matrices are quaternion valued [13] i.e.

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \;\; \gamma^j = \begin{bmatrix} 0 & i\,e_j \\ -i\,e_j & 0 \end{bmatrix} \; (\forall j = 1,2,3; \;\; i = \sqrt{-1}). \tag{19}$$

These γ matrices satisfy the following relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \quad -2 \quad g_{\mu\nu} \tag{20}$$

where

$$g_{\mu\nu} \quad = \quad (-1, +1, +1, +1) \ (\forall \ \mu, \ \nu = 0, 1, 2, 3). \tag{21}$$

Let us write the Dirac equation in terms of a quaternion valued spinor ψ. Now multiplying equation (16) by quaternion basis element e_2 , adding the resultant to equation (15) and using equation (10) , we get the Dirac equation as

$$i\gamma^\mu(\partial_\mu\psi) - m\psi \quad = \quad 0. \tag{22}$$

Similarly, we may write the quaternion conjugate Dirac equation as

$$i(\partial_\mu\overline{\psi})\gamma^\mu + m\overline{\psi} \quad = \quad 0. \tag{23}$$

Dirac equations (22-23) provide the four current as

$$j^\mu \quad = \quad \overline{\psi}\,\gamma^\mu\psi \tag{24}$$

which satisfies the continuity equation $\partial_\mu j^\mu = 0$.

4 Quaternionic $SU(2)$ Global gauge symmetry

In global gauge symmetry , the unitary transformations are independent of space and time. Accordingly, under $SU(2)$ global gauge symmetry, the quaternion spinor ψ transforms as

$$\psi \quad \longmapsto \psi' = \quad U\,\psi \tag{25}$$

where U is 2×2 unitary matrix and satisfies

$$U^{\dagger}U \quad = UU^{\dagger} = UU^{-1} = U^{-1}U = \quad 1. \tag{26}$$

On the other hand, the quaternion conjugate spinor transforms as

$$\overline{\psi} \longmapsto \overline{\psi^{'}} \quad = \quad \overline{\psi}U^{-1} \tag{27}$$

and hence the combination $\psi\overline{\psi} = \overline{\psi}\psi = \psi\overline{\psi^{'}} = \overline{\psi^{'}}\psi$ is an invariant quantity. We may thus write any unitary matrix as

$$U \quad = \quad \exp\left(i\,\hat{H}\right) \tag{28}$$

where H is Hermitian $H^{\dagger} = H$. Thus, we may express the Hermitian 2×2 matrix in terms of four real numbers, $a_{1,}a_{2}$, a_{3}, and θ as

$$\hat{H} \quad = \quad \theta 1 + \sigma_{j}a_{j} = \theta 1 + ie_{j}a_{j} \tag{29}$$

where 1 is the 2×2 unit matrix, σ_{j} are well known 2×2 Pauli-spin matrices and e_{1}, e_{2}, e_{3} are the quaternion units which are connected with Pauli-spin matrices as

$$e_{0} = \quad 1; \qquad e_{j} = -i\sigma_{j}. \tag{30}$$

Hence, we write the Hermitian matrix H as

$$H \quad = \quad \left(\begin{array}{cc} \theta + a_{3} & a_{1} - ia_{2} \\ a_{1} + ia_{2} & \theta - a_{3} \end{array}\right). \tag{31}$$

Equation (28) may now be reduced as

$$U \quad = \quad \exp\left(i\,\theta\right).\exp\left(-e_{j}a_{j}\right). \tag{32}$$

For $SU(2)$ global gauge transformations both θ and $\overrightarrow{a}$ are independent of space time. Here $\exp\left(i\,\theta\right)$ describes the $U(1)$ gauge transformation while the term $\exp\left(-e_{j}a_{j}\right)$ represents the non-Abelian $SU(2)$ gauge transformations. Thus under global $SU(2)$ gauge transformations, the Dirac spinor ψ transforms as

$$\psi \longmapsto \psi^{'} = U\,\psi = \exp\left(-e_{j}a_{j}\right)\psi. \tag{33}$$

The generators of this group e_i obey the commutation relation;

$$[e_j, e_k] = 2f_{jkl}e_l \tag{34}$$

which implies $e_ie_j \neq e_je_i$ showing that the elements of the group are not commutating giving rise to the non abelian gauge structure. So, the partial derivative of spinor ψaccordingly transforms as

$$\partial_\mu\psi(x) \longmapsto \partial_\mu\psi^{'}(x) = \exp\left(-e_{j}a_{j}\right)(\partial_\mu\psi). \tag{35}$$

As such the Lagrangian density is invariant under $SU(2)$ global gauge transformations i.e. $\delta L = 0$. The Lagrangian density thus yields the continuity equation after taking the variations and the definitions of Euler Lagrange equations as

$$\partial_\mu\left\{\frac{\partial L}{\partial(\partial_\mu\psi)}e_k\psi\right\} = \partial_\mu\left\{i\overline{\psi}\gamma^\mu e_k\psi\right\} = \partial_\mu(j^\mu)^k = 0\ (\forall\, k = 1, 2, 3) \tag{36}$$

where the $SU(2)$ gauge current is defined as

$$(j^\mu)^k = \left\{i\overline{\psi}\gamma^\mu e_k\psi\right\}. \tag{37}$$

which is the global current of the fermion field.

5 Quaternionic $SU(2)$ Local Gauge Symmetry

For $SU(2)$ local gauge transformation we may replace the unitary gauge transformation as space- time depaendent. So replacing U by S in equation (25), we get

$$\psi \longmapsto \psi^{'} = S\,\psi \tag{38}$$

in which

$$S = \exp[-\sum_{j} q\, e_j \zeta_j(x)] \tag{39}$$

where parameter $\overrightarrow{\zeta} = -\frac{\overrightarrow{a}(x)}{q}$ with $\overrightarrow{a}(x)$ is infinitesimal quantity depending on space and time and q is described as the coupling constant. Consequently, the Lagrangian density (13) is no more invariant under $SU(2)$ local gauge symmetry as the partial derivative picks an extra term i.e.

$$\partial_\mu \psi \mapsto S\partial_\mu \psi \quad + (\partial_\mu S)\, \psi \quad = D_\mu \psi \tag{40}$$

where the covariant derivative D_μ has been defined in terms of two $Q-$ gauge fields i.e

$$D_\mu \psi \quad = \partial_\mu \psi \quad + A_\mu \psi + B_\mu \psi \tag{41}$$

where $A_\mu = -iA_\mu^j \sigma_j = A_\mu^j e_j = \overrightarrow{A_\mu} \cdot \overrightarrow{e}$ and $B_\mu = -iB_\mu^j \sigma_j = B_\mu^j e_j = \overrightarrow{B_\mu} \cdot \overrightarrow{e}$. Two gauge fields A_μ and $B_\mu$are respectively associated with electric and magnetic charges of dyons (i.e particles carrying the simultaneous existence of electric and magnetic charges). Thus the gauge field $\{A_\mu\}$ is coupled with the electric charge while the gauge field $\{B_\mu\}$ is coupled with the magnetic charge (i.e. magentic monopole). These two gauge fields are subjected by the following gauge transformations

$$A'_\mu \longmapsto S\, A_\mu\, S^{-1} + (\partial_\mu S)\, S^{-1}; \quad B'_\mu \longmapsto S\, B_\mu\, S^{-1} + (\partial_\mu S)\, S^{-1}. \tag{42}$$

For the limiting case of infinitesimal transformations of ζ , we may expand S by keeping only first order terms as

$$S \cong 1 + \overrightarrow{e}.\overrightarrow{a}(x); \quad S^{-1} \cong 1 - \overrightarrow{e}.\overrightarrow{a}(x); \quad \partial_\mu (S) \cong \overrightarrow{e}.\partial_\mu \{\overrightarrow{a}(x)\}. \tag{43}$$

So, on replacing partial derivative of global gauge symmetry to covariant derivative of local gauge symmetry, we may write the invariant Lagrangian density for the quaternion $SU(2)$ gauge fields in the following form

$$L = i\overline{\psi}\gamma_\mu (D_\mu \psi) - m\overline{\psi}\psi, \tag{44}$$

which yields the following current densities of electric and magnetic charges of dyons i.e

$$J_{\mu}=(j_{\mu})_{electric}+(j_{\mu})_{magnetic}=i\mathsf{e}\overline{\psi}\gamma_{\mu}\psi+i\mathsf{g}\overline{\psi}\gamma_{\mu}\psi. \tag{45}$$

where e is the electric charge and g is the magnetic charge. Equation (45) does not satisfy the usual continuity equation i.e. $\partial^{\mu}J_{\mu}\neq 0$ but satisfies the Noetherian form of continuity equation with covariant derivative as

$$D_{\mu}J^{\mu}=0. \tag{46}$$

6 Definition of Octonions

An octonion x is expressed [15] as a set of eight real numbers

$$x=e_{0}x_{0}+e_{1}x_{1}+e_{2}x_{2}+e_{3}x_{3}+e_{4}x_{4}+e_{5}x_{5}+e_{6}x_{6}+e_{7}x_{7}=e_{0}x_{0}+\sum_{A=1}^{7}e_{A}x_{A} \tag{47}$$

where $e_{A}(A=1,2,...,7)$ are imaginary octonion units and e_{0} is the multiplicative unit element. Set of octets $(e_{0},\,e_{1},\,e_{2},e_{3},e_{4},e_{5},e_{6},e_{7})$ are known as the octonion basis elements and satisfy the following multiplication rules

$$e_{0}=1;\;\;e_{0}e_{A}=e_{A}e_{0}=e_{A};\quad e_{A}e_{B}=-\delta_{AB}e_{0}+f_{ABC}e_{C}.\;(A,B,C=1,2,.....,7) \tag{48}$$

The structure constants f_{ABC} is completely antisymmetric and takes the value 1 for following combinations,

$$f_{ABC}=\;\;+1;\;\;\forall(ABC)=(123),\,(471),\,(257),\,(165),\,(624),\,(543),\,(736). \tag{49}$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra $\mathcal{O}$ is described over the algebra of real numbers having the vector space of dimension 8. As such we may write the following relations among octonion basis elements

$$\begin{aligned} [e_A,\ e_B] &= 2\, f_{ABC}\, e_C; \\ \{e_A,\ e_B\} &= -2\,\delta_{AB} e_0; \\ e_A(\, e_B\, e_C) &\neq (e_A\, e_B\,)\, e_C \end{aligned} \tag{50}$$

where brackets [,] and { , } are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol. Octonion conjugate is defined as

$$\overline{x} = e_0x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7 = e_0x_0 - \sum_{A=1}^{7} e_Ax_A \tag{51}$$

where we have used the conjugates of basis elements as $\overline{e_0} = e_0$ and $\overline{e_A} = -e_A$. Hence an octonion can be decomposed in terms of its scalar $(Sc(x))$ and vector $(Vec(x))$ parts as

$$Sc(x) = \frac{1}{2}(\,x + \overline{x}\,); \quad Vec(x) = \frac{1}{2}(\,x - \overline{x}\,) = \sum_{A=1}^{7} e_Ax_A. \tag{52}$$

Conjugates of product of two octonions and its own are described as

$$\overline{(x\, y)} = \overline{y}\,\overline{x}; \qquad \overline{(\overline{x})} = x. \tag{53}$$

while the scalar product of two octonions is defined as

$$\langle x\,,\, y\rangle = \tfrac{1}{2}(x\,\overline{y} + y\,\overline{x}) = \tfrac{1}{2}(\overline{x}\, y + \overline{y}\, x) = \sum_{\alpha=0}^{7} x_\alpha\, y_\alpha. \tag{54}$$

The norm $N(x)$ and inverse x^{-1}(for a nonzero x) of an octonion are respectively defined as

$$\begin{aligned} N(x) = x\,\overline{x} = \overline{x}\, x &= \sum_{\alpha=0}^{7} x_\alpha^2 . e_0; \\ x^{-1} &= \frac{\overline{x}}{N(x)} \Longrightarrow x\, x^{-1} = x^{-1}\, x = 1. \end{aligned} \tag{55}$$

The norm $N(x)$ of an octonion x is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(x\,y) = \quad N(x)\,N(y) = \quad N(y)\,N(x). \tag{56}$$

Equation (50) shows that octonions are not associative in nature and thus do not form the group in their usual form. Non - associativity of octonion algebra $\mathcal{O}$ is provided by the associator $(x, y, z) = (xy)z - x(yz)\ \forall x, y, z \in \mathcal{O}$ defined for any three octonions. If the associator is totally antisymmetric for exchanges of any three variables, i.e. $(x, y, z) = -(z, y, x) = -(y, x, z) = -(x, z, y)$, then the algebra is called alternative. Hence, the octonion algebra is neither commutative nor associative but, is alternative.

7 Gellmann λ matrices

In order to extend the symmetry from $SU(2)$ to $SU(3)$ we replace three Pauli spin matrices by eight Gellmann λ matrices. $\lambda_j \ \ (j = 1, 2, ...8)$ be the 3×3 traceless Hermitian matrices introduced by Gell-Mann.Their explicit forms are;

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \tag{57}$$

which satisfy the following properties as

$$\begin{aligned}
(\lambda_j)^{\dagger} =& \lambda_j; \\
Tr\lambda_j =& 0; \\
Tr\left(\lambda_j\lambda_k\right) =& 2\delta_{jk}; \\
[\lambda_j, \lambda_k] =& 2F_{jkl}\lambda_l \ (\forall \ j, k, l = 1, 2, 3, 4, 5, 6, 7, 8);
\end{aligned} \tag{58}$$

where F_{jkl} are the structure constants of $SU(3)$ group defined as

$$F_{123} = 1; \ F_{147} = F_{257} = F_{435} = F_{651} = F_{637} = \frac{1}{2}; \ F_{458} = F_{678} = \sqrt{\frac{3}{2}}. \tag{59}$$

8 Relation between Octonion and Gellmann Matrices

Let us establish the relationship between octonion basis elements e_A and Gellmann λ matrices. Comparing equations (50) and (58), we get

$$F_{ABC} \ = \ f_{ABC} \ (\forall \, ABC = 123) \tag{60}$$

and

$$F_{ABC} = \frac{1}{2} f_{ABC} \ (\forall \, ABC = 147, 246, 257, 435, 516, 637). \tag{61}$$

Equation (60) leads to

$$\frac{[e_A, e_B]}{[\lambda_A, \lambda_B]} = \frac{e_C}{i\lambda_C} \ (\forall \, A, B, C = 1, 2, 3) \Rightarrow [e_A, e_B] = [\lambda_A, \lambda_B] \ \ (\forall \, e_C = i\lambda_C). \tag{62}$$

On the other hand equation (61) gives rise to

$$\begin{aligned}
\frac{[e_A, e_B]}{[\lambda_A, \lambda_B]} =& \frac{e_C}{2i\lambda_C} \ (\forall A, B, C = 516, 624, 471, 435, 673, 572) \\
\Rightarrow [e_A, e_B] =& [\lambda_A, \lambda_B] \ \ (\forall \, e_C = i\frac{\lambda_C}{2}).
\end{aligned} \tag{63}$$

We may now describe the correspondence between the matrix λ_8 and octonion units in the following manner i.e.

$$\lambda_8 \Longrightarrow -\frac{2}{i\sqrt{3}}\left\{[e_4,e_5]+[e_6,e_7]\right\} \Longrightarrow -\frac{2}{i\sqrt{3}}\left(e_4e_5-e_5e_4+e_6e_7-e_7e_6\right) \Longrightarrow \frac{8e_3}{i\sqrt{3}}. \tag{64}$$

Hence we may describe one to mapping (interrelationship) between octonion basis elements and Gellmann λ matrices by using equations (62-63) as,

$$e_A \propto \lambda_A \Rightarrow e_A = k\lambda_A \tag{65}$$

where k is proportionality constant depending on the different values of A i.e.$k = i\,(\forall\, A = 1,2,3)$ and $k = \frac{i}{2}\,(\forall\, ABC = 516, 624, 471, 435, 673, 572)$. From equation (64), we also get

$$e_3 \propto \lambda_8 \Rightarrow e_3 = \mathtt{k}\lambda_8 \tag{66}$$

where $\mathtt{k} = \frac{i\sqrt{3}}{8}$.With these relations between the octonion units and Gellmann λ matrices, we may develop the octonion quantum chromodynamics in consistent way. To do this, let us establish the following commutation relations for octonion basis elements and Gellmann λ matrices i.e.

$$\begin{aligned}
\frac{[e_6,e_5]}{[\lambda_6,\lambda_5]} &= \frac{[e_4,e_7]}{[\lambda_4,\lambda_7]} = \frac{e_1}{2i\lambda_1} = \frac{1}{2k_1} \Longrightarrow \lambda_1 = -ie_1k_1;\\
\frac{[e_4,e_6]}{[\lambda_4,\lambda_6]} &= \frac{[e_5,e_7]}{[\lambda_5,\lambda_7]} = \frac{e_2}{2i\lambda_2} = \frac{1}{2k_2} \Longrightarrow \lambda_2 = -ie_2k_2;\\
\frac{[e_5,e_4]}{[\lambda_5,\lambda_4]} &= \frac{[e_6,e_7]}{[\lambda_6,\lambda_7]} = \frac{e_3}{2i\lambda_3} = \frac{1}{2k_3} \Longrightarrow \lambda_3 = -ie_3k_3;\\
\frac{[e_7,e_1]}{[\lambda_7,\lambda_1]} &= \frac{[e_6,e_2]}{[\lambda_6,\lambda_2]} = \frac{[e_3,e_5]}{[\lambda_3,\lambda_5]} = \frac{e_4}{2i\lambda_4} = \frac{1}{2k_4} \Longrightarrow \lambda_4 = -ie_4k_4;\\
\frac{[e_4,e_3]}{[\lambda_4,\lambda_3]} &= \frac{[e_7,e_2]}{[\lambda_7,\lambda_2]} = \frac{[e_1,e_6]}{[\lambda_1,\lambda_6]} = \frac{e_5}{2i\lambda_5} = \frac{1}{2k_5} \Longrightarrow \lambda_5 = -ie_5k_5;\\
\frac{[e_5,e_1]}{[\lambda_5,\lambda_1]} &= \frac{[e_7,e_3]}{[\lambda_7,\lambda_3]} = \frac{[e_2,e_4]}{[\lambda_2,\lambda_4]} = \frac{e_6}{2i\lambda_6} = \frac{1}{2k_6} \Longrightarrow \lambda_6 = -ie_6k_6;\\
\frac{[e_1,e_4]}{[\lambda_1,\lambda_4]} &= \frac{[e_2,e_5]}{[\lambda_2,\lambda_5]} = \frac{[e_3,e_6]}{[\lambda_3,\lambda_6]} = \frac{e_7}{2i\lambda_7} = \frac{1}{2k_7} \Longrightarrow \lambda_7 = -ie_7k_7.
\end{aligned} \tag{67}$$

As such, we may get the following relationship between Gell Mann λmatrices and octonion units:

$$\lambda_A = -\,i\,e_A k_A \;(\forall A = 1,2,3,4,5,6,7);$$
$$\lambda_8 = -\,i\,\frac{8}{\sqrt{3}}\,e_3. \tag{68}$$

9 Octonionic Reformulation of QCD

The local gauge theory of color $SU(3)$group gives the theory of QCD. The QCD (quantum chromodynamics) is very close to Yang-Mills (non Abelian) gauge theory. The above mentioned $SU(2)$ gauge symmetry describes the symmetry of the weak interactions. On the other hand, the theory of strong interactions,quantum chromodynamics (QCD), is based on colour $SU(3)$ (namely $SU(3)_c$) group. This is a group which acts on the colour indices of quark favours described in the form of a basic triplet i.e.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \rightarrow \begin{pmatrix} R \\ B \\ G \end{pmatrix} \tag{69}$$

where indices R, B, and G are the three colour of quark flavours. Under $SU(3)_c$symmetry, the spinor ψ transforms as

$$\psi \longmapsto \psi' = U\psi = \exp\left\{ i\lambda_a \alpha^a(x)\right\}\psi \tag{70}$$

where λ are Gellmann matrices , $a = 1, 2, ...8$ and the parameter α is space time dependent. We may develop accordingly the octonionic reformulation of quantum chromodynamics (QCD) on replacing the Gellmann λ matrices by octonion basis elements e_A given by equations (65) and (66). Now calculating the value of $\lambda_a \alpha^a(x) = \sum_{a=1}^{8} \lambda_a \alpha^a(x)$ and using the relations between GellMann λ matrices and octonion units given by equations (**68**), we find

$$\sum_{a=1}^{8} \lambda_a \alpha^a(x) = -ie_1 k_1 \alpha^1(x) - ie_2 k_2 \alpha^2(x) - ie_3\left(k_3 \alpha^3(x) + k_8 \alpha^8(x)\right)$$
$$-ie_4 k_4 \alpha^4(x) - ie_5 k_5 \alpha^5(x) - ie_6 k_6 \alpha^6(x) - ie_7 k_7 \alpha^7(x). \tag{71}$$

Now taking following transformations

$$k_1\alpha_1 \longmapsto \beta^1\,;\; k_2\alpha_2 \longmapsto \beta^2\,;\; (k_3\alpha_3 + k_8\alpha_8) \longmapsto \beta^3;$$
$$k_4\alpha_4 \longmapsto \beta^4\,;\; k_5\alpha_5 \longmapsto \beta^5\,;\; k_6\alpha_6 \longmapsto \beta^6\,;\; k_7\alpha_7 \longmapsto \beta^7; \tag{72}$$

we get

$$\sum_{a=1}^{8} \lambda_a \alpha^a\,(x) = -ie_1\beta^1 - ie_2\beta^2 - ie_3\beta^3 - ie_4\beta^4 - ie_5\beta^5 - ie_6\beta^6 - ie_7\beta^7. \tag{73}$$

It may also be written in the following generalized compact form i.e.

$$\sum_{a=1}^{8} \lambda_a \alpha^a\,(x) = -i\sum_{q=1}^{7} e_q \beta^q\,(x)\,; \tag{74}$$

which may be written in terms of the following traceless Hermitian matrix form as

$$-\,i\sum_{q=1}^{7} e_q \beta^q\,(x) = \begin{bmatrix} \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -\alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}. \tag{75}$$

Now (69) becomes

$$\psi \longmapsto \psi^{'} = U\psi = \exp\left\{e_q\beta^q(x)\right\} \tag{76}$$

So we may write the locally gauge invariant $SU(3)_c$,Lagrangian density in the following form;

$$L_{local} = \;\left(i\overline{\psi}\gamma_\mu D_\mu\psi - m\overline{\psi}\psi\right)\; -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \tag{77}$$

where

$$D_\mu\psi \;=\; \partial_\mu\psi + \boldsymbol{e}\,e_a A^a_\mu\psi + \mathbf{g}\,e_a B^a_\mu\psi \tag{78}$$

and

$$\begin{aligned} G^a_{\mu\nu} =& \partial_\mu V^a_\nu - \partial_\nu V^a_\mu - \mathbf{q}\,f_{abc}V^b_\mu V^c_\nu \\ =& \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \mathbf{e}\,f_{abc}A^b_\mu A^c_\nu\right) + \left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu - \mathbf{g}\,f_{abc}B^b_\mu B^c_\nu\right). \end{aligned} \tag{79}$$

Here in equations (78) and (79), the **e** and **g** are the coupling constants due to the occurence of respectively the electric and magnetic charges on dyons. On the similar ground the two gauge fields $\{A_\mu\}$and $\{B_\mu\}$ are present in the theory due to the occurence of respectively the electric and magnetic charges on dyons. As such, in the present theory we have two kinds of color gauge groups respectively associated with the two gauge fields of electric and magnetic charges on dyons. Hence the locally gauge covariant Lagrangian density is written as

$$L_{local} = \left(i\overline{\psi}\gamma_\mu\partial_\mu\psi - m\overline{\psi}\psi\right) - \mathbf{e}\,\left(\overline{\psi}\gamma^\mu\psi\right)e_a A_\mu^a - \mathbf{g}\,\left(\overline{\psi}\gamma^\mu\psi\right)e_a B_\mu^a - \frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} \quad (80)$$

which leads to the folowing expression for the gauge covariant current density of coloured dyons

$$J_\mu^a = \mathbf{e}\,\left(\overline{\psi}\gamma^\mu\psi\right)e_a + \mathbf{g}\,\left(\overline{\psi}\gamma^\mu\psi\right)e_a. \quad (81)$$

which leads to the conservation of Noetherian current in octonion formulation of $SU(3)_c$ gauge theory of quantum chromodynamics (QCD) i.e.

$$D_\mu J^\mu = 0 \quad (82)$$

where $J^\mu = J^{\mu a}\lambda_a$.

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